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On the shape of a gas bubble in a viscous extensional flow

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The method developed previously (Youngren & Acrivos 1975) for obtaining numerical solutions to the Stokes equations for flows past solid particles is extended to problems with free boundaries. This technique is applied to the determination of steady shapes for an inviscid gas bubble symmetrically placed in an extensional flow. For large surface tension the computed bubble shape is found to be in excellent agreement with that obtained analytically by Barthès-Biesel & Acrivos (1973), while for small surface tension it agrees with an expression derived by Buckmaster (1972) using slender-body theory.

1. Introduction

In a recent communication (Youngren & Acrivos 1975, henceforth referred to as I), the problem of obtaining the solution of the Stokes equations for flow past an arbitrary solid particle was reduced to that of solving a system of integral equations of the first kind for a distribution of Stokeslets on the particle surface. First the validity and accuracy of the approach were demonstrated by comparing the numerical results with known analytic solutions and then new solutions were computed for several problems involving finite cylinders. However, many interesting and significant systems in creeping flow involve liquid drops and gas bubbles which are deformable and whose shape, being unknown *a priori*, must be determined as part of the solution.

Techniques for dealing with such systems have been presented by Richardson (1968), who considered the case of an inviscid two-dimensional bubble freely suspended in a hyperbolic or in a simple shear flow, and by Buckmaster & Flaherty (1973), who dealt with the corresponding problem of a two-dimensional drop freely suspended in a hyperbolic flow of a liquid having the same viscosity. However, since both studies were based on the use of complex-variable theory, their approach cannot be extended to the more realistic three-dimensional case, and hence it would be useful to adapt the general method of I to these so-called free-boundary problems. This appears to be feasible, because equation (2.9) in I applies whether the surface is rigid or not and in fact can be viewed as consisting of three equations which relate the three components of the stress force on the surface. To be sure, when, as in I, the surface is rigid, the no-slip boundary

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condition reduces these relations to a system of three integral equations of the first kind for the three unknown components of the stress force; however, the development certainly applies to more general boundary conditions. For example, in the case of an inviscid gas bubble, all three stress force components and one velocity component on the surface are known and the integral relations (I 2.9) become three integral equations for the two unknown velocity components and the steady shape. The exact details for implementing this method will now be described and the technique will be illustrated by obtaining a numerical solution to the problem of finding the shape of an inviscid gas bubble symmetrically placed in an extensional flow.

This example is, of course, part of the more general problem of determining theoretically the steady shape of an individual drop or bubble freely suspended in an unbounded shear flow of another fluid, a subject of considerable interest to the field of emulsion rheology. Unfortunately, the problem is so difficult that analytic solutions are possible only in very special cases. Thus, when the Reynolds number is sufficiently small for the creeping-flow equations to apply, Taylor (1932, 1934), Cox (1969), Frankel & Acrivos (1970) and Barthès-Biesel & Acrivos (1973) constructed solutions for slightly non-spherical drops and bubbles by expanding the unknown shape in terms of its deformation from sphericity. Within this context, the special case of an inviscid gas bubble symmetrically placed in an extensional flow has been considered in detail because of its relative simplicity. Steady shapes were obtained by Frankel & Acrivos (1970) and by Barthès-Biesel & Acrivos (1973) for values of k, the ratio of surface tension to viscous forces, greater than approximately 10, but steady solutions could not be found by their method for smaller k. At the other extreme, i.e. when k is very small, the bubble becomes elongated and its shape was given analytically by Buckmaster (1972), who used slender-body theory with a distribution of sources and Stokeslets along the bubble axis. Surprisingly, Buckmaster found a multiplicity of steady shapes for each (small) value of k, the non-uniqueness resulting from the fact that a parameter appears in his solution which can assume an infinite number of discrete values. Apart from their non-uniqueness, these slender-bubble solutions are also open to question because they may not correspond to a branch that is physically attainable when the bubble is progressively elongated from its initial spherical shape following a steady increase in the strength of the applied shear. Thus there exists a need to determine steady shapes for intermediate k and, if possible, to ascertain whether some of the slender shapes that are predicted from Buckmaster's analysis can be attained at small k. This aim will be accomplished by adapting the method of I to this problem.

2. Problem statement and method of solution

Creeping extensional flow past an incompressible† inviscid gas bubble (figure 1) is described by

$$\partial^2 v_i/\partial x_j \partial x_j = \partial P/\partial x_i, \quad \partial v_j/\partial x_j = 0, \quad \mathbf{x} \in \Omega, \tag{2.1}$$

† The method of solution for a compressible bubble is briefly discussed following (2.10).

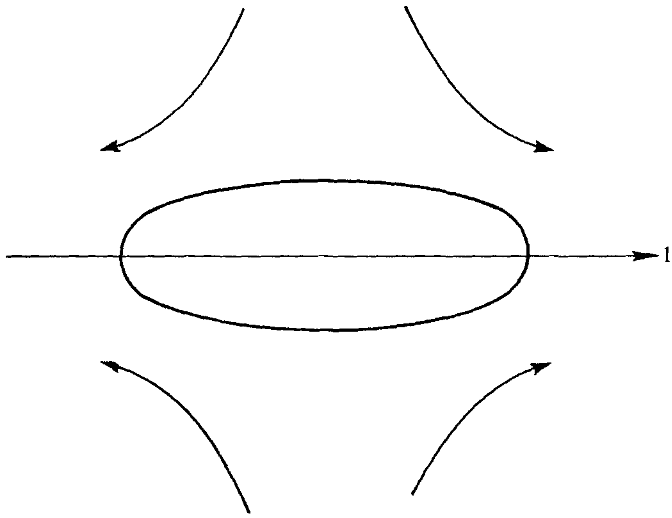

FIGURE 1. Gas bubble symmetrically placed in an extensional flow.

$$\left.\begin{array}{l} P\rightarrow 0, \quad v_i \rightarrow GC_{ij}x_j \equiv V_i \quad \text{as} \quad (x_j x_j)^{\frac{1}{2}} \rightarrow \infty, \\ C_{ij} = 2\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2} - \delta_{i3}\delta_{j3}, \\ t_{ij}n_j + P^* n_i = -n_i \gamma\, \partial n_j/\partial x_j, \quad v_j n_j = 0 \quad \text{on} \quad S, \end{array}\right\} \tag{2.2}$$

together with the requirement that V, the bubble volume, should remain constant. Cartesian tensor notation is employed, v_i is the fluid velocity, P and P^* are the liquid and gas pressures respectively, $t_{ij} = -P\delta_{ij} + \mu(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ is the stress tensor for the liquid, n_i is the *inward* unit normal to the gas–liquid interface S, G is the shear strength of the undisturbed flow, μ is the liquid viscosity and γ is the surface tension. Introducing the disturbance velocity $u_i \equiv v_i - V_i$ and non-dimensionalizing all distances by a, the radius of the spherical bubble in the absence of any shear, all velocities by Ga, the stress tensor in the liquid phase by $G\mu$ and the gas pressure by γ/a, (2.1) and (2.2) become

$$\partial^2 u_i/\partial x_j \partial x_j = \partial P/\partial x_i, \quad \partial u_j/\partial x_j = 0, \quad \mathbf{x} \in \Omega, \tag{2.3}$$

$$u_i \rightarrow 0, \quad P \rightarrow 0 \quad \text{as} \quad (x_j x_j)^{\frac{1}{2}} \rightarrow \infty, \tag{2.4a}$$

$$\left.\begin{array}{l} u_j n_j = -C_{ij}x_j n_i \qquad\qquad\qquad\qquad (2.4b) \\ \sigma_{ij}n_j = -kn_i(\partial n_j/\partial x_j + P^*) - 2C_{ij}n_j \quad (2.4c) \end{array}\right\} \quad \text{on} \quad S,$$

where the dimensionless parameter $k \equiv \gamma/\mu G a$ and

$$\sigma_{ij} \equiv -P\delta_{ij} + \partial u_i/\partial x_j + \partial u_j/\partial x_i.$$

Also, we need to satisfy the condition $V = \frac{4}{3}\pi$, where V is the dimensionless bubble volume.

As shown in I, the velocity u_i and the stress force $f_i = \sigma_{ij}n_j$ on S are related by

$$u_i(\mathbf{x}) + \frac{3}{2\pi}\iint_S \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)\, n_j u_k}{r_{xy}^5}\, dS_y = \frac{1}{4\pi}\iint_S \left[\frac{\delta_{ij}}{r_{xy}} + \frac{(x_i - y_i)(x_j - y_j)}{r_{xy}^3}\right] f_j\, dS_y, \quad \mathbf{x} \in S, \tag{2.5}$$

where $\mathbf{x}$ and $\mathbf{y}$ are position vectors, integration over S is with respect to $\mathbf{y}$, and $r_{xy} \equiv |\mathbf{x}-\mathbf{y}|$. At first glance it might appear that (2.5), which can be written symbolically as

$$u_i(x) + \iint_S K_{ij}(x,y)\, u_j(y)\, dS_y = \iint_S F_i(x,y)\, dS_y, \tag{2.6}$$

could be solved as follows: assume a bubble shape, compute F_i from (2.4c), determine u_i by solving numerically (2.6), which now becomes an integral equation of the second kind, and then adjust the shape of the bubble until the kinematic condition (2.4b) is satisfied everywhere on S. Unfortunately, though, this apparently straightforward procedure is difficult to implement for the following reason. As shown by Ladyzhenskaya (1963, p. 59), the adjoint to the homogeneous part of (2.6),

$$u_i(x) + \iint_S K_{ji}(y,x)\, u_j(y)\, dS_y = 0, \tag{2.7}$$

has a single eigensolution, equal to $n_i(x)$, and therefore the homogeneous part of (2.6) has also a single eigensolution $u_i^{(e)}$. It can easily be shown of course that n_i, the eigensolution of (2.7), is orthogonal to the right-hand side of (2.6) because

$$\iint_S n_i(x)\, F_i(x,y)\, dS_x = 0,$$

and hence a particular solution $u_i^{(p)}$ to (2.6) does indeed exist. Nevertheless, since the complete solution to (2.6) will contain an unknown multiple of $u_i^{(e)}$, the procedure just outlined would be cumbersome from the practical point of view.

An alternative approach was therefore selected which did not suffer from the deficiencies described above. First, in view of the axial symmetry of the present problem, it is convenient to transform to cylindrical co-ordinates with the 1-direction parallel to the axis of symmetry and to express u_i and f_i in terms of their normal components u_n and f_n and tangential components u_t and f_t (with $u_t > 0$ if $u_1 > 0$):

$$u_t = u_r n_1 - u_1 n_r, \quad u_n = u_1 n_1 + u_r n_r.$$

Here, and in what follows, the tensor summation convention is abandoned and the subscript r denotes the radial component in cylindrical co-ordinates. Because of the axial symmetry, (2.5) can be integrated analytically in the azimuthal direction to yield, after rearrangement,

$$\int_{-l}^{l} K_1(x,y,R)\, u_t(y)\, dy + n_r(x)\, u_t(x) = \int_{-l}^{l} K_2 f_n\, dy + n_1 u_n + \int_{-l}^{l} K_3 u_n\, dy + \int_{-l}^{l} K_4 f_t\, dy \tag{2.8a}$$

and
$$\int_{-l}^{l} K_5 u_t\, dy - n_1 u_t = \int_{-l}^{l} K_6 f_n\, dy + n_r u_n + \int_{-l}^{l} K_7 u_n\, dy + \int_{-l}^{l} K_8 f_t\, dy. \tag{2.8b}$$

The kernels K are given by Youngren (1975) and are available from the authors on request. Also x and y are the 1-components of $\mathbf{x}$ and $\mathbf{y}$, $R(x)$ is the unknown radius of S and l is defined by $R(\pm l) = 0$. The requirement of constant bubble volume becomes

$$\int_0^l R^2(x)\, dx = \tfrac{2}{3}, \tag{2.9a}$$

while the boundary conditions (2.4) reduce to

$$\left.\begin{aligned} u_n &= -2n_1 x + n_r R \\ f_n &= k\left[\frac{1}{R(1+R'^2)^{\frac{1}{2}}} - \frac{R''}{(1+R'^2)^{\frac{3}{2}}} - P^*\right] - 2(3n_1^2-1) \\ f_t &= 6n_1 n_r \end{aligned}\right\} \quad \text{on} \quad S, \tag{2.9b}$$

which are to be substituted into the right-hand sides of (2.8a, b). A solution to the above system can then be obtained by assuming an initial $R(x)$, determining the kernels K in (2.8) plus u_n, f_n and f_t [the latter three quantities from (2.9b)], solving the two integral equations of the second kind (2.8a, b) independently for u_t as in I, and then adjusting $R(x)$ until the two functions u_t thus computed agree with each other to within a specified relative error. It is worth noting that the condition (2.9a) can be disregarded during this iteration because, once a converged solution has been computed, the solution to the original problem, denoted here by a caret, can be recovered simply using the transformation $(\hat{k}, \hat{R}, \hat{x}) = \alpha^{-1}(k, R, x)$, where

$$\alpha \equiv \left\{\frac{3}{2}\int_0^l R^2(x)\,dx\right\}^{\frac{1}{3}}.$$

Of course, this method will run into difficulties if the homogeneous parts of (2.8a, b) have an eigensolution which is the same for both equations. This will be the case if $n_i u_i^{(e)} \equiv 0$, where, as before, $u_i^{(e)}$ refers to the eigensolution of the homogeneous part of (2.6). Noting though that for the sphere and for the long circular cylinder $u_i^{(e)} = n_i$, we can take it for granted that $u_i^{(e)}$ will not be everywhere orthogonal to n_i, except perhaps for bodies of very unusual geometry; hence it should be possible to obtain a solution of (2.8) subject to (2.9) using the scheme just outlined.

It should also be mentioned at this point that, as can be easily verified, any constant scalar multiple of n_j, when added to f_j, does not contribute to the right-hand side of (2.5) since the corresponding term vanishes upon integration with respect to dS_y over S. Consequently, the gas pressure P^* in (2.4c) and (2.9b) can be set arbitrarily, e.g. equal to zero, without affecting the bubble shape $R(x)$ as computed from the procedure just described. For an incompressible bubble, the value of P^* is evidently of little interest but, if required, it can be determined, once a converged bubble shape $R(x)$ has been computed, using, for example, the condition that

$$-\iint_S \sigma_{ij} n_j u_i^{(e)} dS \equiv \iint_S \left\{k n_i \left[\frac{\partial n_j}{\partial x_j} + P^*\right] + 2C_{ij} n_j\right\} u_i^{(e)} dS = -\int_S u_i n_i dS = 0, \tag{2.10}$$

where, again, $u_i^{(e)}$ is the eigensolution of the homogeneous part of (2.6). The above follows directly from the reciprocal theorem (Happel & Brenner 1965, p. 85) and the fact that n_i is the surface stress force corresponding to the eigensolution $u_i^{(e)}$. Equation (2.10) can also be used together with an equation of state, e.g. the ideal-gas law $P^* V =$ constant, to determine P^* for a compressible bubble. In the latter case, P^* is of course an integral part of the solution since its value is required to specify the volume V.

In the computational scheme just outlined, the iterative determination of R was readily accomplished using Newton's method to calculate systematically successive approximations $R^{(n)}$ to R. Specifically, denote the collocation points of the numerical scheme described in I by x_m ($m = 1, 2, \ldots, N$), define $h_m \equiv R(x_m)$ and let $\hat{u}_m^{(n)}$ and $\tilde{u}_m^{(n)}$ be the solutions of (2.8a, b) respectively at x_m calculated in iteration n. Then expand

$$\hat{u}_i^{(n+1)} = \hat{u}_i^{(n)} + (\partial\hat{u}_i/\partial h_j)^{(n)}[h_j^{(n+1)} - h_j^{(n)}] + O[(h_j^{(n+1)} - h_j^{(n)})^2],$$

with a similar expansion for $\tilde{u}_i^{(n+1)}$. Summation over j from 1 to N is implied. But convergence is achieved when $\hat{u}_i^{(n+1)} = \tilde{u}_i^{(n+1)}$, which implies that at convergence

$$0 = \hat{u}_i^{(n)} - \tilde{u}_i^{(n)} + [h_j^{(n+1)} - h_j^{(n)}](\partial\hat{u}_i/\partial h_j - \partial\tilde{u}_i/\partial h_j)^{(n)} + O[(h_j^{(n+1)} - h_j^{(n)})^2], \tag{2.11}$$

or

$$h_j^{(n+1)} = h_j^{(n)} + \{A_{ij}^{(n)}\}^{-1}(\tilde{u}_i - \hat{u}_i)^{(n)} + O[(h_j^{(n+1)} - h_j^{(n)})^2], \tag{2.12}$$

where $\{A_{ij}^{(n)}\} = \{(\partial\hat{u}_i/\partial h_j - \partial\tilde{u}_i/\partial h_j)^{(n)}\}$ is referred to as the Jacobian matrix. The Jacobian was calculated numerically by perturbing $h_j^{(n)}$ by $\Delta h_j^{(n)}$ and then solving (2.8a, b) for $\hat{u}_i^{(n)} + \Delta\hat{u}_i^{(n)}$ and $\tilde{u}_i^{(n)} + \Delta\tilde{u}_i^{(n)}$ to yield

$$A_{ij}^{(n)} = \left[\frac{\Delta\hat{u}_i - \Delta\tilde{u}_i}{\Delta h_j}\right]^{(n)}.$$

Early attempts to obtain solutions for $k = 5$ with a spheroidal shape as the initial guess $R^{(1)}$ all failed, thereby indicating that the radius of convergence of Newton's method as implemented here is not very large, a conclusion that was also reached during the course of obtaining the solutions to be described in §3. Thus the iteration was carried out starting with a sphere, which is the shape of a gas bubble in the limit of infinite k, as the initial guess for $k = 20$. The converged solution for each k was then used as the initial guess for the next smaller k. This method of obtaining solutions is natural, and corresponds to the course of a hypothetical experiment in which, beginning with a sphere in a stagnant fluid, the bubble is progressively extended by incrementally increasing the strength of the extensional flow.

Convergence was accelerated by under-relaxing as follows: (i) define the error

$$E_i \equiv |A_{ij}^{(n)}(h_j^{(n+1)} - h_j^{(n)}) + \hat{u}_i^{(n)} - \tilde{u}_i^{(n)}|,$$

which is the absolute value of the right-hand side of (2.11); (ii) rather than attempting to drive the error to zero everywhere using (2.12), under-relax by accepting a relative error

$$E_i^{(n)} = (1-\omega)\,|\tilde{u}_i^{(n)} - \hat{u}_i^{(n)}|, \quad 0 < \omega \leqslant 1,$$

and (iii) minimize

$$W^{(n+1)} \equiv (h_j^{(n+1)} - h_j^{(n)})(h_j^{(n+1)} - h_j^{(n)}), \tag{2.13a}$$

a measure of the change in shape at iteration $n+1$, subject to the constraints

$$-E_i^{(n)} \leqslant A_{ij}^{(n)}(h_j^{(n+1)} - h_j^{(n)}) + \hat{u}_i^{(n)} - \tilde{u}_i^{(n)} \leqslant E_i^{(n)}. \tag{2.13b}$$

This provides for non-uniform under-relaxation and prevents sudden, drastic changes in shape. This was found to be advantageous since once a shape $R^{(n+1)}$

was produced which was not smooth (as a result of one or several of the quantities $(h_j^{(n+1)}-h_j^{(n)})/h_j^{(n)}$ being large) the procedure would immediately diverge. Note that when the relaxation factor ω is unity the constraints are tight and no relaxation occurs, while as $\omega \to 0$ the solution to (2.13*a*, *b*) is $R^{(n+1)} \to R^{(n)}$.

The fact that the objective function $W^{(n)}$ of the minimum problem (2.13*a*) is quadratic with linear constraints (2.13*b*) suggests that the powerful techniques of quadratic programming (Wilde & Beightler 1967, p. 69) could be applied to this problem. However, since the purpose of this work was not to obtain an accurate solution to (2.13*a*, *b*) (which would be used only once and then discarded), a simple efficient direct climbing technique (Wilde & Beightler 1967, p. 271) was employed to approximate the solution to the minimum problem.

Since the numerical method assumes that $R^{(n)}$ is known for $-l \leqslant x \leqslant l$ and since (2.9*b*) requires that $R'^{(n)}$ and $R''^{(n)}$ be available for $-l \leqslant x \leqslant l$, the method of interpolating $R^{(n)}$ in terms of $h_j^{(n)}$ is quite important. A natural choice is to use spline functions since the method of solving (2.8*a*, *b*) divides the interval $[-l, l]$ into N subintervals, which is exactly what is done by splines. Cubic splines, which render $R''^{(n)}$ continuous, were employed using standard methods (Reinsch 1967).

3. Results

The final bubble shapes are sketched in figure 2, where $x_1 = 0$ is a plane of symmetry and the 1-axis is the axis of symmetry. The calculated shapes for $k \lesssim 15{\cdot}0$ are in excellent agreement with the results of Frankel & Acrivos (1970) for large k while, as shown in figure 2, those for $k \lesssim 4$ agree very well with one of the shapes given by Buckmaster's (1972) slender-body analysis, which is thus seen to be physically attainable. In all cases $R^{(n)}$ was considered to have converged if $W^{(n)} < 10^{-8}/\omega$. The optimum value of ω was found to decrease from $0{\cdot}9$ for $k = 15$ to $0{\cdot}5$ for $k = 4$. Also a small N, typically 8, was used during early iterations but was eventually increased to 14 for the final determination of R. For efficiency, since the Jacobian matrix changed only slightly with each iteration and since the converged solution for k was used as the initial guess for the next lower k, the Jacobian for the higher k was employed in all iterations for the new k until convergence was attained, and the Jacobian was then updated (however, in one case the Jacobian was updated before convergence in order to accelerate the process).

As is customary in considering shapes of drops or bubbles, the deformation $D \equiv (l-R(0))/(l+R(0))$ is shown in figure 3, which illustrates the excellent agreement of the $O(k^{-1})$ analysis of Frankel & Acrivos (1970) and the $O(k^{-2})$ theory of Barthès-Biesel & Acrivos (1973) with the numerical results at large k.

Thus it seems clear that there are steady solutions for all k down to $k = 4$ for an inviscid gas bubble in an extensional flow. This is in agreement with Buckmaster's (1972) analysis and contradicts the prediction of Barthès-Biesel & Acrivos (1973) that inviscid drops will burst if the strain is too large. As conjectured by these authors, the failure of the perturbation methods that apply for large k to predict steady solutions for k less than a certain value probably

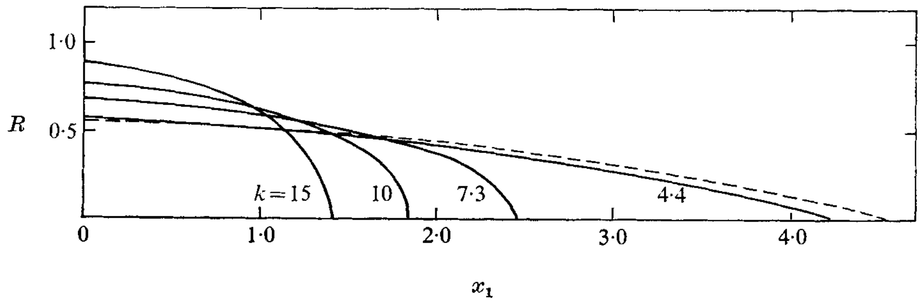

FIGURE 2. Steady shapes of an inviscid gas bubble symmetrically placed in an extensional flow. ——, numerical solution; – – –, slender-body theory (Buckmaster 1972, $n = 2$, $k = 4{\cdot}4$).

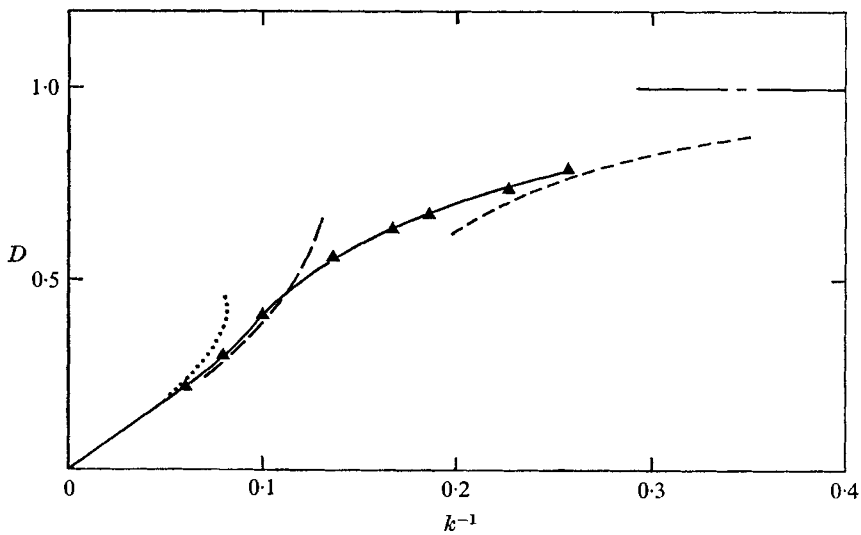

FIGURE 3. Deformation of an inviscid gas bubble in an extensional flow. —▲—, numerical solution; · · · · ·, $O(k^{-1})$ theory (Frankel & Acrivos 1970); — — —, $O(k^{-2})$ theory (Barthès-Biesel & Acrivos 1973); – – –, slender-body theory (Buckmaster 1972, $n = 2$); — – —, limiting deformation.

results from one's inability to represent a bubble shape which is quite non-spherical in terms of a series containing a small number of surface spherical harmonics. At any rate, though, it is evident that the predictions made by Barthès-Biesel & Acrivos (1973) concerning the bursting of drops of low viscosity should be treated with extreme caution, not only because of the failure of their theory for inviscid bubbles in an extensional flow, but also because, as shown in their paper, the agreement between their analysis and the available experimental data for hyperbolic and simple shear flows becomes progressively worse as λ, the ratio of the drop viscosity to that of the external fluid, is decreased.

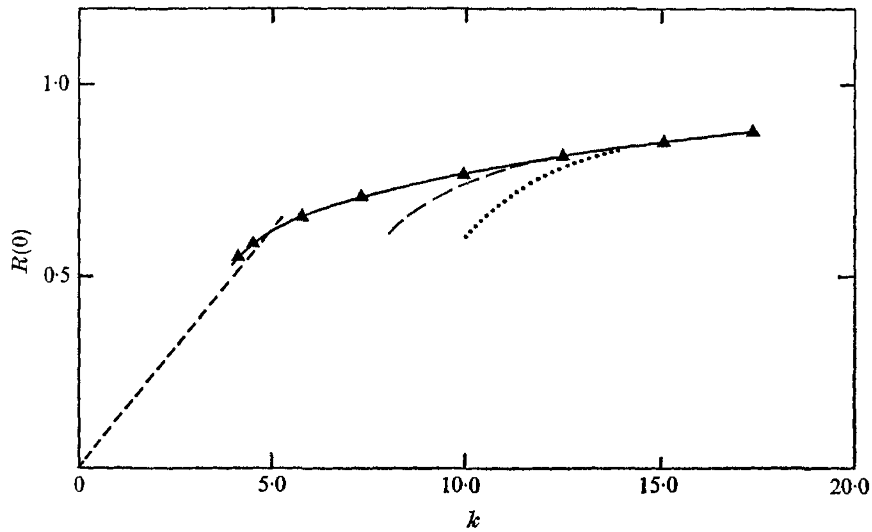

FIGURE 4. Maximum radius of a gas bubble in an extensional flow. —▲—, numerical solution; · · · · ·, $O(k^{-1})$ theory (Frankel & Acrivos 1970); — — —, $O(k^{-2})$ theory (Barthès-Biesel & Acrivos 1973); – – –, slender-body theory (Buckmaster 1972, $n = 2$).

Figure 4 presents $R(0)$ as a function of k and supports the observation from figure 2 that the present results are consistent with the solution with $n = 2$ in Buckmaster's (1972) set of asymptotic solutions as $k \to 0$:

$$R(x) = \frac{k}{4n}\left[1-\left(\frac{x}{l}\right)^n\right], \quad n = 2, 4, 6, \ldots. \tag{3.1}$$

Note that, for a fixed V, the solution with $n = 2$ has the smallest ratio $l/R(0)$, i.e. the smallest deformation D. Therefore, if one assumes that all the solutions (3.1) are stable and physically attainable, the $n = 2$ shape would be the one to be realized in the hypothetical experiment described earlier in which the strength of the shear rate is incrementally increased from zero. Since the numerical procedure is analogous to this hypothetical experiment, it is to be expected that the present analysis would converge to Buckmaster's $n = 2$ solution.

Another reason why our solution, as $k \to 0$, should conform to (3.1) with $n = 2$ is that Buckmaster's (1972) result, in our notation, $P^* = 4(n+1)/k$ with $n = 2$ can also be derived directly from his equation (2.22) by requiring that $R(z)$ be analytic and even with $R''(0) \neq 0$. Similarly, his corresponding expression for P^* with $n = 4$ implies that $R''(0) = 0$ and $R^{\text{iv}}(0) \neq 0$. In our numerical scheme, though, in which the bubble was progressively elongated starting from a spherical shape, $R''(0)$ was always found to be negative and the use of cubic splines ensured that $R''(z)$ and, in all likelihood, at least some of the higher derivatives remained continuous everywhere. Hence the agreement of our results with the $n = 2$ solution of Buckmaster should have been expected once it was shown that our numerical scheme led to slender bubbles for small enough k.

Unfortunately, for increasingly slender bubble shapes it became more and more difficult to obtain converged solutions because of a decreasing radius of convergence for Newton's method and the need for more under-relaxation (smaller ω). Thus no attempt was made to verify whether the increasingly slender shapes associated with $n = 4, 6, 8, \ldots$, in (3.1) could be generated by the present method. Consequently, although the present study has established the fact that the $n = 2$ solution of (3.1) is physically attainable, the question of non-uniqueness raised by Buckmaster (1972) remains unresolved.

Also, Buckmaster sought to determine whether the ends of the bubble are locally rounded or cusp-shaped and, using a local analysis, found that multiple cusp-shaped solutions which are valid within an exponentially small distance from the ends are a possibility. Unfortunately, as implemented, the present method can yield only the gross bubble shape and is unable to resolve such extremely fine local details.

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